VALENCE AND OSCILLATION OF FUNCTIONS IN THE UNIT DISK

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Abstract. We investigate the number of times that nontrivial solutions of equations $u'' + p(z)u = 0$ in the unit disk can vanish—or, equivalently, the number of times that solutions of $S(f) = 2p(z)$ can attain their values—given a restriction $|p(z)| \leq b(|z|)$. We establish a bound for that number when b satisfies a Nehari-type condition, identify perturbations of the condition that allow the number to be infinite, and compare those results with their analogs for real equations $\varphi'' + q(t)\varphi = 0$ in $(-1, 1)$.

This paper investigates the number of times that nontrivial solutions of an equation $u'' + p(z)u = 0$ in the unit disk $\mathbb{D} \subseteq \mathbb{C}$ can vanish. Which conditions $|p(z)| \leq b(|z|)$ imply that the number of zeroes is finite? In terms of b, how many zeroes can there be? And how do the answers to those questions compare with what happens with equations $\varphi'' + q(t)\varphi = 0$ for real-valued functions in $(-1, 1)$?

The results for the complex setting are equivalent to statements about the valence of a locally injective, meromorphic mapping f in $\mathbb{D}$ whose Schwarzian derivative $S(f) = (f''/f')' - \frac{1}{2}(f''/f')^2$ satisfies a bound $|Sf(z)| \leq 2b(|z|)$. Because every solution of $S(f) = 2p$ is a quotient of linearly independent solutions of $u'' + pu = 0$, its valence

$$\sup_{c \in \mathbb{C} \cup \{\infty\}} \#\big\{z \in \mathbb{D} : f(z) = c\big\},$$

equals the oscillation number

$$\sup_{\text{solutions } u \not\equiv 0} \#\big\{z \in \mathbb{D} : u(z) = 0\big\},$$

of that equation. In particular, both quantities are finite or both infinite.

The equation $u'' + pu = 0$ in $\mathbb{D}$ has finite oscillation number if p is bounded. Indeed, in view of Sturm's theorem below and the standard method summarized in

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(i) of Theorem 10 (see Section 1), a bound $|p| \le C$ implies that any two zeroes of a nontrivial solution are at least $\pi/\sqrt{C}$ units apart. Boundedness, however, is not a necessary condition. Using a method of Nehari [11], B. Schwarz [14] has shown that finite oscillation occurs if $|p(z)| \le 1/(1-|z|^2)^2$ for all z near $\partial\mathbb{D}$. His theorem complements an observation by Hille [8] that, when $c > 1$, some nontrivial solutions of $u'' + c(1-z^2)^{-2}u = 0$ have infinitely many zeroes in $\mathbb{D}$.

The first of the main results in this paper is a quantitative version of Schwarz's theorem. For a holomorphic function p in $\mathbb{D}$, let

$$M_p(r) = \max\{|p(z)| : |z| = r\}, \qquad r \in [0,1).$$

Theorem 1. *There are constants A and B such that, if $p : \mathbb{D} \to \mathbb{C}$ is holomorphic and $|p(z)| \le 1/(1-|z|^2)^2$ whenever $R \le |z| < 1$, then nontrivial solutions of $u'' + pu = 0$ satisfy*

$$\#\{z \in \mathbb{D} : u(z) = 0\} \le \frac{A}{1-R} + B\int_0^R \frac{\sqrt{M_p(r)}}{1-r}\,dr.$$

Although based on simple principles, Theorem 1 often provides satisfactory estimates for the maximal oscillation number among the equations in a family defined by a condition $|p(z)| \le b(|z|)$. It provides the upper bound in the following situation, for example, and that is of the correct order of magnitude:

Theorem 2. *Let $\alpha \in (0,1)$, and for $C \ge 0$ let $N_\alpha(C)$ be the maximum of the oscillation numbers among the equations $u''+pu = 0$ in which $|p(z)| \le C/(1-|z|^2)^{2\alpha}$ for all $z \in \mathbb{D}$. Then there are positive numbers k_α, K_α, and A_α such that*

$$k_\alpha C^{1/(2-2\alpha)} \le N_\alpha(C) \le K_\alpha C^{1/(2-2\alpha)}, \qquad C \ge A_\alpha.$$

Theorem 1 also implies that the maximum $N_0(C)$ for the family defined by the condition $|p| \le C$ is $O(\sqrt{C}\log C)$. That maximum, however, might be $O(\sqrt{C})$. It is at least $2\sqrt{C}/\pi$, as one sees from equations $u'' + Cu = 0$, and in Section 3 we show that it actually exceeds $k\sqrt{C}$ for some $k > 2/\pi$ when C is large. Bounds $N_{1/2}(C) = O(C\log C)$ and $N_0(C) = O(C)$ were established in [2].

Analogs of Theorems 1 and 2 for equations $\varphi'' + q(t)\varphi = 0$ in $(-1,1)$ in which φ and q are real-valued take a somewhat different form. The analysis in that situation rests upon the following:

Sturm Comparison Theorem [7]**.** *Let $q \le Q$ be continuous functions in $[a,b]$, and let φ and ψ be solutions of $\varphi'' + q\varphi = 0$ and $\psi'' + Q\psi = 0$, respectively, with φ having no zero in (a,b). If $\varphi(a) = \varphi(b) = 0$, then ψ has a zero in (a,b) unless $q = Q$ and ψ is a multiple of φ. The same conclusion holds if $\varphi(b) = 0$, $\varphi(a)$ and $\psi(a)$ are nonzero, and $(\psi'/\psi)(a) \le (\varphi'/\varphi)(a)$, or if $\varphi(a) = 0$, $\varphi(b)$ and $\psi(b)$ are nonzero, and $(\psi'/\psi)(b) \ge (\varphi'/\varphi)(b)$.*

As with complex equations, we define the oscillation number of a real equation $\varphi'' + q\varphi = 0$ in an interval I, where q is continuous, to be the supremum $N \in \{1, 2, \ldots, \infty\}$ of the number of zeroes of nontrivial solutions. Every nontrivial solution then vanishes N or $N-1$ times in I, for Sturm's theorem implies that the zeroes of any two such solutions are either identical or interlaced in a strictly alternating pattern. Equations for which $N = 1$ are said to be *disconjugate* in I, and those for which $N = \infty$ are said to be *oscillatory* there. The equation $\varphi'' + c(1-t^2)^{-2}\varphi = 0$, for example, is disconjugate in $(-1,1)$ when $c \le 1$ and oscillatory when $c > 1$, as one sees from the general solution

$$\varphi(t) = \begin{cases} (1-t^2)^{1/2}\big\{\alpha\cosh(\delta L(t)/2) + \beta\sinh(\delta L(t)/2)\big\} & \text{if } c = 1-\delta^2 < 1 \\ (1-t^2)^{1/2}\big\{\alpha + \beta L(t)\big\} & \text{if } c = 1 \\ (1-t^2)^{1/2}\big\{\alpha\cos(\delta L(t)/2) + \beta\sin(\delta L(t)/2)\big\} & \text{if } c = 1+\delta^2 > 1 \end{cases}$$

with $L(t) = \log((1+t)/(1-t))$.

Sturm's theorem shows that it is not so much $|q|$ as the signed quantity q that matters in estimating the oscillation number of a real equation $\varphi'' + q\varphi = 0$, for larger coefficients yield larger or identical oscillation numbers. Another phenomenon is the sensitivity of the oscillation number N to "spikes" in q. For a constant-coefficient $\varphi'' + C\varphi = 0$ in an interval of length ℓ,

$$-1 + (\ell/\pi)\sqrt{\max\{C,0\}} \le N \le 1 + (\ell/\pi)\sqrt{\max\{C,0\}}\,. \qquad (1)$$

In view of the ability to approximate an arbitrary continuous function from above and below by step functions, Sturm's theorem then suggests the estimate

$$N \approx \frac{1}{\pi}\int_I \sqrt{q(t)^+}\,dt, \qquad x^+ = \max\{x, 0\},$$

as a rule of thumb in the general case. One needs hypotheses, however, that prevent the graph of q from having many sharp spikes, for when that happens N can be considerably larger than the integral. The effect is severe enough to preclude analogs of Theorem 1 with an integral involving $(q^+)^{1/2}$ or $|q|^{1/2}$. In particular, Theorem 6 in Section 1 shows that no bound of the form $N \le A_a + B_a \int_{-a}^{a} |q(t)|^{1/2}\,dt$ holds for equations in which q is supported in an interval $[-a, a] \subseteq (-1, 1)$. The analog we give uses $(M_q^+)^{1/2}$ instead, where

$$M_q(r) = \max\bigl\{q(t) : |t| \le r\bigr\}, \qquad r \in [0, 1).$$

This is the smallest nondecreasing function b such that $q(t) \le b(|t|)$ for all $t \in [0, 1)$.

Theorem 3. *If $q : (-1, 1) \to \mathbb{R}$ is continuous and $q(t) \le 1/(1 - t^2)^2$ whenever $R \le |t| < 1$, then nontrivial real solutions of $\varphi'' + q\varphi = 0$ satisfy*

$$\#\bigl\{t \in (-1, 1) : \varphi(t) = 0\bigr\} \le 3 + \frac{4}{\pi}\int_0^R \sqrt{M_q(r)^+}\,dr.$$

The proof also provides the bound $1 + (4/\pi)\int_0^1 (M_q(r)^+)^{1/2}\,dr$ for all q.

Theorem 2 addresses the rate of growth, as $C \to \infty$, of the maximal oscillation number $N(C)$ of complex equations $u'' + pu = 0$ in $\mathbb{D}$ that satisfy certain conditions $|p(z)| \le Cb(|z|)$, and it shows that the rate of growth can depend on b. One can study the same issue for real equations $\varphi'' + q\varphi = 0$ in $(-1, 1)$, but there $N(C)$ is usually asymptotic to a constant times $\sqrt{C}$. To state the result, it is enough to consider equations $\varphi'' + Cq\varphi = 0$ with q fixed, for Sturm's theorem implies that, among equations whose coefficients are bounded by $Cb(|t|)$, the equation $\varphi'' + Cb(|t|)\varphi = 0$ itself has maximal oscillation number.

Theorem 4. *If $q : (-1, 1) \to \mathbb{R}$ is continuous and $(M_q^+)^{1/2}$ is integrable, then the oscillation numbers of the equations $\varphi'' + Cq\varphi = 0$ satisfy the asymptotic relation*

$$N(C) \sim \frac{\sqrt{C}}{\pi}\int_{-1}^{1} \sqrt{q(t)^+}\,dt \qquad \textit{as } C \to \infty.$$

The hypothesis is equivalent to the assumption that $q(t) \le b(|t|)$ for some nondecreasing function b in $[0,1)$ with $(b^+)^{1/2}$ integrable.

In both the real and complex settings, bounds $c/(1-|x|^2)^2$ on the coefficient for $|x|$ near one imply a finite oscillation number exactly when $c \le 1$. That might lead one to wonder if every bound $b(|x|)$ that implies finite oscillation for real equations also does so for complex equations (as might (i) in Theorem 10). The answer is no. Real equations in $(-1,1)$ have finite oscillation number if

$$\big|q(t)\big| \le \frac{1 + \big\{\log(1-|t|)\big\}^{-2}}{(1-t^2)^2}\,, \qquad |t| \approx \pm 1;$$

see Theorem 1 in [3] or Exercise 1.2 in Chapter XI of [7]. In $\mathbb{D}$, however, every bound $|p(z)| \le \beta(|z|)/(1-|z|^2)^2$ whose numerator decays to one at a slower-than-linear rate as $|z| \to 1$ allows infinite oscillation:

Theorem 5. *If $\beta : [0,1) \to (0,\infty)$ is continuous and $\lim_{r\to 1}(\beta(r)-1)/(1-r) = \infty$, then there is a holomorphic function p in $\mathbb{D}$ satisfying $|p(z)| \le \beta(|z|)/(1-|z|^2)^2$ for all $z \in \mathbb{D}$ such that some nontrivial solution of $u'' + pu = 0$ has infinitely many zeroes.*

To complement this theorem, it would be desirable to show that conditions

$$\big|p(z)\big| \le \frac{1 + C\big(1-|z|\big)}{(1-|z|^2)^2}\,, \qquad z \in \mathbb{D},$$

with $C > 0$ imply finite oscillation. We do not know whether they do, however.

Section 1 of this paper addresses oscillation in the real setting; Theorems 3 and 4 and the assertions surrounding them emerge from stronger results proved there. Section 2 treats a way in which equations $u'' + pu = 0$ transform to equations of the same form under a change of independent variable. Using the transform, we identify perturbations of the Nehari bound that allow equations with oscillation number at least two and construct equations $u'' + pu = 0$ in $\mathbb{D}$ with large oscillation number. Section 3 contains the proofs of Theorems 1 and 2 and Section 4 the proof of Theorem 5.

1. Oscillation numbers of real equations

This section treats differential equations $\varphi''+q(t)\varphi=0$ in which q is a continuous, real-valued function in an interval $I\subseteq\mathbb{R}$ (of any kind) and, implicitly, φ is real-valued.

As mentioned in the introduction, "spikes" in the graph of q can cause such an equation to have large oscillation number relative to the integral of $(q^+)^{1/2}$. The following theorem demonstrates that:

Theorem 6. *For every closed interval $[a,b]$, positive integer N, and $\epsilon>0$, there is a continuous, nonnegative function q in $\mathbb{R}$, supported in $[a,b]$ and satisfying $\int_a^b q(t)^{1/2}\,dt<\epsilon$, such that some nontrivial solution of $\varphi''+q\varphi=0$ vanishes more than N times in $[a,b]$.*

Proof. With $(b-a)/N=2\delta$, it is enough to produce a continuous, nonnegative function r, supported in $[-\delta,\delta]$ and satisfying $\int_{-\delta}^{\delta} r(t)^{1/2}\,dt<\epsilon/N$, such that some nontrivial solution of $\varphi''+r\varphi=0$ vanishes at both δ and $-\delta$; the assertions in the theorem will then hold for the function $q(t)=\sum_{k=1}^N r(t-a-(2k-1)\delta)$, for nontrivial solutions of $\varphi''+q\varphi=0$ that vanish at a will also vanish at $a+2\delta,\cdots,a+2N\delta=b$.

Let $f(t)=\eta(1-t^2)^+$, where $\eta>0$ is small enough that $\int_{-1}^1 f(t)^{1/2}dt<\epsilon/N$ and that the solution $\psi=\psi_\eta$ of $\psi''+f\psi=0$ with $\psi(0)=1$ and $\psi'(0)=0$ remains positive throughout $[0,1]$; the latter is possible since, by continuous dependence of solutions upon parameters, $\psi_\eta(t)\to 1$ as $\eta\to 0$, the convergence being uniform on bounded sets. This solution vanishes at a point $\tau>1$, for it is linear in $[1,\infty)$ with derivative $\psi'(1)=-\int_0^1 f(t)\psi(t)\,dt<0$; being even, ψ also vanishes at $-\tau$. The function $\varphi(t)=\psi(t\tau/\delta)$ then vanishes at both δ and $-\delta$ and solves an equation $\varphi''+r\varphi=0$ in which the coefficient function $r(t)=(\tau/\delta)^2 f(t\tau/\delta)$ is supported in $[-\delta,\delta]$ and satisfies $\int_{-\delta}^{\delta} r(t)^{1/2}dt=\int_{-z}^{z} f(u)^{1/2}du<\epsilon/N$. □

One way to control this phenomenon is to require that q^+ be bounded by a piecewise-monotonic function whose square root is integrable. The key observation is the following:

Lemma 7. *Let $a < b$ be successive zeroes of a nontrivial solution of $\varphi'' + q\varphi = 0$, where q is continuous and nonnegative in $[a, b]$, and let $c \in (a, b)$ be a critical point of φ. If q is nondecreasing in $[c, b]$, then $\int_c^b q(t)^{1/2}\,dt \geq \pi/2$; if q is nonincreasing in $[a, c]$, then $\int_a^c q(t)^{1/2}\,dt \geq \pi/2$.*

Proof. One may assume that $\varphi > 0$ in (a, b), and also that q is nondecreasing in $[c, b]$, for the function $\psi(t) = \varphi(-t)$ satisfies $\psi'' + q(-t)\psi = 0$. Under those assumptions, suppose that $\int_c^b q(t)^{1/2}\,dt < \pi/2$, and let

$$w(x) = \cos\biggl(\int_c^x \sqrt{q(t)}\,dt\biggr)\varphi'(x) + \sin\biggl(\int_c^x \sqrt{q(t)}\,dt\biggr)\sqrt{q(x)}\cdot\varphi(x), \qquad x \in [c, b].$$

This function is continuous, and a computation shows that the derivate

$$\bigl(D_* w\bigr)(x) := \liminf_{h\to 0} \frac{w(x+h) - w(x)}{h} \in [-\infty, \infty]$$

satisfies

$$\bigl(D_* w\bigr)(x) = \sin\left(\int_c^x q(t)^{1/2}\,dt\right)\cdot\bigl(D_*\sqrt{q}\,\bigr)(x)\cdot\varphi(x) \geq 0, \qquad x \in (c, b).$$

By a bisection argument, one concludes that w is nondecreasing in each interval $[c', b'] \subseteq (c, b)$ and hence, by continuity, in $[c, b]$. But that is false, for $w(c) = 0$ and $w(b) = \cos(\int_c^b q(t)^{1/2}dt)\varphi'(b) < 0$, and the lemma follows. □

In the situation of this lemma, the integral of $\sqrt{q}$ over the remaining part of $[a, b]$ can be arbitrarily small, with the integral over $[a, b]$ exceeding $\pi/2$ by an arbitrarily small amount, even if q is monotonic. Indeed, given $\epsilon > 0$, let $q(t) = e^{t/\epsilon}$ for $t < 0$ and $q(t) = 1$ thereafter. The solution of $\varphi'' + q\varphi = 0$ with $\varphi(0) = 1$ and $\varphi'(0) = 0$ is given by $\varphi(t) = \cos t$ when $t \geq 0$, and it has a zero in $(-\infty, 0)$ since $\varphi'' < 0$ whenever $\varphi > 0$. The hypotheses of Lemma 7 then hold with a equal to the largest zero of φ in $(-\infty, 0)$ and $b = \pi/2$, but $\int_a^b q(t)^{1/2}\,dt < \epsilon + \pi/2$. Concatenations of such examples also show that the factor $2/\pi$ in the next result cannot be reduced:

Theorem 8. *Let $q : I \to \mathbb{R}$ be continuous. Suppose that I is the union of n pairwise disjoint intervals I_j and that q^+ is bounded by a continuous, monotonic function b_j in each. If b is the union of those functions, then nontrivial solutions* *of* $\varphi'' + q\varphi = 0$ *satisfy*

$$\#\{t \in I : \varphi(t) = 0\} \le n + \frac{2}{\pi}\int_I \sqrt{b(t)}\,dt.$$

Proof. By Lemma 7, the number of zeroes of a nontrivial solution of $\psi'' + b_j\psi = 0$ satisfies $\int_{I_j} b_j(t)^{1/2}\,dt \ge (\pi/2)(N-1)$, or $N \le 1 + (2/\pi)\int_{I_j} b_j(t)^{1/2}\,dt$. Sturm's theorem implies that the same quantity bounds the number of zeroes of nontrivial solutions of $\varphi'' + q\varphi = 0$ in I_j, and the theorem follows by summing over j. □

Theorem 8 yields the bound $2 + (4/\pi)\int_0^1 (M_q(r)^+)^{1/2}\,dr$ for the oscillation number of an equation $\varphi'' + q\varphi = 0$ in $(-1, 1)$. As asserted after Theorem 3 in the introduction, that bound can be reduced by one. It is sufficient to establish the reduced bound for equations in which q is nonnegative, even, and nondecreasing in $[0, 1)$, which is to say that $q(t) = M_q(|t|)^+$, for the general result then follows by a Sturm comparison with $\varphi'' + M_q(|t|)^+\varphi = 0$. Under those assumptions, let $t_1 < \cdots < t_N$ be successive zeroes of a nontrivial solution φ. If $t_N < 0$ or $t_1 \ge 0$, then Theorem 8 implies that $N \le 1 + (2/\pi)\int_{t_1}^{t_N} q(t)^{1/2}\,dt$, and that quantity is less than or equal to the asserted bound. Suppose, then, that $t_k < 0 \le t_{k+1}$ for some k, and let c be a critical point of φ in (t_k, t_{k+1}). By Lemma 7, the integral of $\sqrt{q}$ over $[c, t_{k+1}]$ is at least $\pi/2$ if $c \ge 0$, and that over $[t_k, c]$ is so if $c < 0$. Thus the integral over $[t_k, t_{k+1}]$ is at least $\pi/2$ in either case, and by Theorem 8

$$\begin{aligned} N &\le 1 + \frac{2}{\pi}\int_{t_1}^{t_k} \sqrt{q(t)}\,dt + 1 + \frac{2}{\pi}\int_{t_{k+1}}^{t_N} \sqrt{q(t)}\,dt \\ &\le 2 + \frac{2}{\pi}\left(\int_{-1}^{1} \sqrt{q(t)}\,dt - \frac{\pi}{2}\right) = 1 + \frac{4}{\pi}\int_0^1 \sqrt{q(t)}\,dt. \end{aligned}$$

Similarly, nontrivial solutions φ have at most $1 + (4/\pi)\int_0^R (M_q(r)^+)^{1/2}\,dr$ zeroes in $[-R, R]$ when $R < 1$. If in addition $q(t) \le 1/(1-t^2)^2$ whenever $R \le |t| < 1$, then a Sturm comparison with the equation $\psi'' + (1-t^2)^{-2}\psi = 0$ discussed in the introduction shows that such solutions have at most one zero in each of $(-1, -R)$ and $(R, 1)$ and hence at most $3 + (4/\pi)\int_0^R (M_q(r)^+)^{1/2}\,dr$ zeroes overall. This argument proves Theorem 3.

Theorem 4 is a consequence of the following:

Theorem 9. *Suppose that $q : I \to \mathbb{R}$ is continuous and that, in each of the at most two components of the complement $I - J$ of a closed, bounded subinterval J, the function q^+ is bounded by a continuous, monotonic function b with $\sqrt{b}$ integrable. If $N(C)$ is the oscillation number of the equation $\varphi'' + Cq\varphi = 0$ in I, then*

$$N(C) \sim \frac{\sqrt{C}}{\pi} \int_I \sqrt{q(t)^+}\, dt \qquad \text{as} \;\; C \to \infty.$$

Proof. Given $\epsilon > 0$, one can expand J so that the integrals of the functions $\sqrt{b}$ in $I - J$ sum to less than ϵ. Nontrivial solutions of $\varphi'' + Cq\varphi = 0$ then have at most $2 + 2\epsilon\sqrt{C}/\pi$ zeroes in $I - J$ by Theorem 8, and hence

$$\limsup_{C\to\infty} \frac{N(C)}{\sqrt{C}/\pi} \le \limsup_{C\to\infty} \frac{2 + 2\epsilon\sqrt{C}/\pi + N_J(C)}{\sqrt{C}/\pi} = 2\epsilon + \limsup_{C\to\infty} \frac{N_J(C)}{\sqrt{C}/\pi}\,,$$

where $N_J(C)$ is the oscillation number of $\varphi'' + Cq\varphi = 0$ in J.

Sturm's theorem and the estimate (1) for constant-coefficient equations show that, if S and S' are lower and upper Riemann sums for $\int_J (q(t)^+)^{1/2}\, dt$, then

$$S \le \liminf_{C\to\infty} \frac{N_J(C)}{\sqrt{C}/\pi} \le \limsup_{C\to\infty} \frac{N_J(C)}{\sqrt{C}/\pi} \le S'.$$

Since this holds for all such sums, both interior quantities equal $\int_J (q(t)^+)^{1/2}\, dt$. Therefore

$$\liminf_{C\to\infty} \frac{N(C)}{\sqrt{C}/\pi} \ge \liminf_{C\to\infty} \frac{N_J(C)}{\sqrt{C}/\pi} = \int_J \sqrt{q(t)^+}\, dt > -\epsilon + \int_I \sqrt{q(t)^+}\, dt,$$
$$\limsup_{C\to\infty} \frac{N(C)}{\sqrt{C}/\pi} \le 2\epsilon + \int_J \sqrt{q(t)^+}\, dt \le 2\epsilon + \int_I \sqrt{q(t)^+}\, dt,$$

and the theorem follows since ϵ was arbitrary. □

We end this section with some basic ways in which real-variable methods can reveal restrictions on how frequently solutions of complex equations can vanish. The sufficiency of condition (i) has been observed in [9] p. 578, [10] p. 293, and [12].

Theorem 10. *Let $t \mapsto z_t$, $t \in [0, T]$, be a parametrization of a segment $J \subseteq \mathbb{C}$ with constant velocity ζ. If p is a holomorphic function in an open set containing J, then nontrivial solutions of $u'' + pu = 0$ vanish at most once in J if the function $P(t) = \zeta^2 p(z_t)$ in $[0, T]$ satisfies any of the following:*

(i) *Nontrivial real solutions of $\varphi'' + \mathrm{Re}(P) \cdot \varphi = 0$ vanish at most once $[0, T]$,*

(ii) $\mathrm{Im}(P)$ *has just finitely many zeroes and is otherwise of one sign, or*

(iii) $\mathrm{Re}(P) \leq \pi^2/T^2 + m \cdot \mathrm{Im}(P)$ *for some* $m \in \mathbb{R}$*, but* $P(t) \not\equiv \pi^2/T^2$.

Proof. Suppose that a nontrivial solution of $u'' + pu = 0$ has two or more zeroes in J. The function $U(t) = u(z_t)$ in $[0, T]$ then satisfies $U'' + PU = 0$ and vanishes at least twice but is not identically zero. We prove the theorem by considering successive zeroes $x < y$ of U and concluding that none of (i), (ii), or (iii) holds.

Writing $U(t) = r(t)e^{i\theta(t)}$ and $P(t) = a(t)+ib(t)$ in the interval (x, y) and equating the real and imaginary parts of $(U'' + PU)e^{-i\theta(t)}$ to zero gives

$$r'' - (\theta')^2 r + ar = 0, \qquad r\theta'' + 2r'\theta' + br = 0. \tag{2}$$

Here r and θ extend to C^2 functions in $[x, y]$ with r and θ' vanishing at the endpoints, as one sees from power-series expansions of u about z_x and z_y.

Since the solution $r(t)$ of the first equation in (2) is nontrivial and vanishes twice in $[x, y]$, Sturm's theorem implies that some nontrivial solution of $\varphi'' + a\varphi = 0$ vanishes at least twice there and hence twice in $[0, T]$; thus (i) fails. The second equation in (2) states that $(r^2\theta')' = -br^2$. Since $r^2\theta'$ vanishes at x and y, it follows that b either changes sign or is identically zero; thus (ii) fails, also. Finally, suppose that the first condition in (iii) holds for some m. We recall Wirtinger's inequality [6], which asserts that if $f : \mathbb{R} \to \mathbb{R}$ is of class C^1 and τ-periodic with $\int_0^\tau f(t)\,dt = 0$, then $\int_0^\tau f'(t)^2 dt \geq (2\pi/\tau)^2 \int_0^\tau f(t)^2 dt$. Applying that result with $\tau = 2(y - x)$ and

$$f(t) = \begin{cases} r(t - k\tau + x) & \text{if } k\tau \leq t \leq (k + 1/2)\tau \\ -r(k\tau - t + x) & \text{if } (k - 1/2)\tau \leq t \leq k\tau, \end{cases} \qquad k \in \mathbb{Z},$$

gives the first inequality in the computation below, where the last step uses the fact that $\int_x^y b(t)r(t)^2\,dt = -\int_x^y (r^2\theta')'(t)\,dt = 0$:

$$\begin{aligned} &\frac{\pi^2}{(y-x)^2} \int_x^y r(t)^2\,dt \leq \int_x^y r'(t)^2\,dt = -\int_x^y r(t)r''(t)\,dt \\ &= \int_x^y \Big(a(t) - \theta'(t)^2\Big) r(t)^2\,dt \leq \int_x^y \left(\frac{\pi^2}{T^2} + mb(t) - \theta'(t)^2\right) r(t)^2\,dt \\ &= \frac{\pi^2}{T^2} \int_x^y r(t)^2\,dt - \int_x^y \theta'(t)^2 r(t)^2\,dt. \end{aligned}$$

Since $r > 0$ throughout (x,y) and $[x,y] \subseteq [0,T]$, it follows that $[x,y] = [0,T]$, $a(t) \equiv \pi^2/T^2 + mb(t)$, and $\theta'(t) \equiv 0$, and the latter implies that $b(t) \equiv 0$. Therefore $P(t) \equiv \pi^2/T^2$, and (iii) fails. □

2. Change of independent variable

This section exploits a way in which equations $u''+pu=0$ transform to equations of the same form, and having the same oscillation number, under a change of independent variable. Let p be a holomorphic function in an open set $D \subseteq \mathbb{C}$ and F a holomorphic bijection from an open set D' onto D. By the chain rule

$$S\big(f\circ g\big)(z) = \big(Sg\big)(z) + g'(z)^2 \cdot Sf\big(g(z)\big) \tag{3}$$

for the Schwarzian derivative, a function f in D satisfies $S(f) = 2p$ if and only if the function $h = f \circ F$ in D' satisfies $S(h) = 2 \cdot \{\frac{1}{2}S(F) + (F')^2(p\circ F)\}$. That transform of the nonlinear equation is reflected in a transform of $u'' + pu = 0$:

Lemma 11. *If $F : D' \to D$ is a holomorphic bijection between open sets in $\mathbb{C}$, then holomorphic functions p and u in D satisfy $u'' + pu = 0$ if and only if the function $v = (F')^{-1/2}(u\circ F)$ in D' satisfies $v'' + Pv = 0$, where $P = \frac{1}{2}S(F) + (F')^2(p\circ F)$. Similar assertions hold for C^3 changes of variable in real equations $\varphi'' + q\varphi = 0$.*

We omit the proof but note that, unless F is linear, the transformed equation $v'' + Pv = 0$ is *not* the one that results from setting $u(z) = v(F^{-1}(z))$.

Transforming an equation $u'' + pu = 0$ in the unit disk by means of a conformal automorphism $T : \mathbb{D} \to \mathbb{D}$ has the effect of redistributing the function

$$[\![p]\!](z) = (1-|z|^2)^2 \cdot \big|p(z)\big|, \qquad z \in \mathbb{D},$$

in that

$$\begin{aligned} [\![P]\!](z) &= (1-|z|^2)^2 \cdot \big|\tfrac{1}{2}\cdot 0 + T'(z)^2\, p\big(T(z)\big)\big| \\ &= (1-|T(z)|^2)^2 \cdot \big|p\big(T(z)\big)\big| = [\![p]\!]\big(T(z)\big), \qquad z \in \mathbb{D}. \end{aligned} \tag{4}$$

This observation, perhaps in the form

$$[\![S(f\circ T)]\!] = [\![S(f)]\!] \circ T, \qquad T \in \mathrm{Aut}(\mathbb{D}), \tag{5}$$

for mappings f, enters into several results in this paper, such as the following:

Theorem 12 (Nehari [11]**).** *Let p be a holomorphic function in $\mathbb{D}$. If $\Gamma \subseteq \mathbb{D}$ is an arc of a circle orthogonal to $\partial\mathbb{D}$ and $[\![p]\!] \le 1$ throughout Γ, then nontrivial solutions of $u'' + pu = 0$ vanish at most once in Γ.*

Proof. We transform the equation $u'' + pu = 0$ by means of a conformal automorphism T of the unit disk that takes a segment I of the real diameter onto Γ. By (4), the new coefficient P satisfies $|P(t)| \le 1/(1-t^2)^2$ for $t \in I$. Since nontrivial real solutions of $\varphi'' + (1-t^2)^{-2}\varphi = 0$ vanish at most once in $(-1, 1)$, as noted in the introduction, part (i) of Theorem 10 implies that nontrivial solutions of $v'' + Pv = 0$ vanish at most once in I. The assertion then follows from Lemma 11. □

Nehari infers that if $[\![p]\!] \le 1$ throughout $\mathbb{D}$ then nontrivial solutions of $u''+pu = 0$ vanish at most once in $\mathbb{D}$ (equivalently, solutions of $S(f) = 2p$ are univalent), for any two points in $\mathbb{D}$ lie on such a curve Γ. It is of some interest that the conclusion fails for every bound $|p(z)| \le 1/(1-|z|^2)^2 + h(|z|)$ in which h is continuous, nonnegative, and not identically zero. To see that, let $g \le h$ be a function with the same properties that extends continuously to $[0, 1]$; one could let $g(t) = h(t)$ throughout some interval $[0, x]$ with x near one, for example, and $g(t) = \min_{[x,t]} h$ thereafter. A routine application of the Weierstrass theorem yields an even, real polynomial f such that $f \le g$ throughout $[0, 1)$ and $\int_0^1 (1 - r^2) f(r)\,dr > 0$. Consider a function $p(z) = 1/(1-z^2)^2 + \epsilon f(z)$ with $\epsilon \in (0, 1]$. Because $\int_0^1 (1-r^2)\epsilon f(r)\,dr > 0$, Theorem 4 of [3] implies that the solution of $u''+pu = 0$ with $u(0) = 1$ and $u'(0) = 0$ vanishes at least twice in $(-1, 1)$. But since f is even and $1/(1 - z^2)^2 = \sum_{k=0}^{\infty}(k + 1)z^{2k}$, the Maclaurin coefficients for p are positive if ϵ is small, and for such a value

$$\big|p(z)\big| \le p\big(|z|\big) = \frac{1}{(1-|z|^2)^2} + \epsilon f(|z|) \le \frac{1}{(1-|z|^2)^2} + h(|z|), \qquad z \in \mathbb{D}.$$

In Section 4, we need bounds $[\![p]\!](z) \le \beta(|z|)$ in which $\beta(r)$ is somewhat less than one for r near one but large enough elsewhere to allow an oscillation number greater than one. The next result provides them.

Lemma 13. *For all sufficiently small $\mu > 0$, the solution of*

$$u''(z) + \frac{1-\mu^2+2\mu(1-z^2)}{(1-z^2)^2}\,u(z) = 0, \qquad u(0) = 1, \quad u'(0) = 0,$$

vanishes at least twice in $(-1, 1)$.

Proof. The displayed conditions define a real initial-value problem in $(-1, 1)$. Under the change of variable $z = \tanh t$, the transform in Lemma 11 yields the problem

$$v''(t) + \bigl(-\mu^2 + 2\mu \operatorname{sech}^2 t\bigr)v(t) = 0, \qquad v(0) = 1, \quad v'(0) = 0,$$

in $\mathbb{R}$, and by symmetry it is enough to show that the solution v_μ vanishes somewhere in $(0, \infty)$ when μ is sufficiently small. We note that $v_0(t) = 1$.

By the variational equations for dependence of solutions upon parameters, the function $w(t) = \partial_\mu(v_\mu(t))|_{\mu=0}$ satisfies $w''(t) + 2\operatorname{sech}^2 t = 0$ with $w(0) = w'(0) = 0$, and $w'(t) = \partial_\mu(v'_\mu(t)|_{\mu=0}$. Solving that initial-value problem, one finds that

$$v_\mu(t) = 1 - 2\mu\log(\cosh t) + o(\mu), \quad v'_\mu(t) = -2\mu\tanh t + o(\mu),$$

as $\mu \to 0$ with t fixed. It follows that $v_\mu(1) > 0$ and $v'_\mu(1)/\mu < -v_\mu(1)$ when $\mu > 0$ is sufficiently small, for $\tanh(1) > 1/2$. But then the solution

$$\varphi(t) = v_\mu(1)\cdot\cosh\bigl(\mu(t-1)\bigr) + \frac{v'_\mu(1)}{\mu}\cdot\sinh\bigl(\mu(t-1)\bigr)$$

of $\varphi'' - \mu^2\varphi = 0$ with $\varphi(1) = v_\mu(1)$ and $\varphi'(1) = v'_\mu(1)$ vanishes somewhere in $(1, \infty)$. By Sturm's theorem, v_μ does, also. □

As a complement to Lemma 13, one can show that the solution of

$$u''(z) + \frac{1-\mu^2+\mu(1-z^2)}{(1-z^2)^2}\,u(z) = 0, \qquad u(0) = 1, \quad u'(0) = 0,$$

does not vanish in $(-1, 1)$ when $\mu > 0$. Thus the factor two in the lemma cannot be replaced with one (although any number greater than one would work).

A final result of this kind will be used in Section 3:

Lemma 14. *There is a positive number $\omega < \pi^2$ such that, for all $k \in \{1, 2, \ldots\}$, the solution of $u'' + \omega(k/2+1)^2 z^k u = 0$, $u(0) = 0$, $u'(0) = 1$, has at least $k + 3$ zeroes in $\mathbb{D}$.*

Proof. In view of the identity $u(e^{2\pi i/(k+2)}z) = e^{2\pi i/(k+2)}u(z)$ implied by uniqueness of solutions, it is enough to show that u vanishes somewhere in $(0,1)$. Let ω be any number between $\pi^2 - 1/4 + 1/9$ and π^2, and consider the real equations

$$\varphi''(t) + \omega(k/2+1)^2 t^k \varphi(t) = 0, \qquad \psi''(s) + \left(\omega + \frac{1/4 - 1/(k+2)^2}{s^2}\right)\psi(s) = 0.$$

These are related, in the sense of Lemma 11, by the change of variable $t = s^{2/(k+2)}$; thus the formula $\psi(s) = \{2/(k+2)\cdot s^{-k/(k+2)}\}^{-1/2}\varphi(s^{2/(k+2)})$ establishes a bijection between the solution sets. Let $\epsilon > 0$ be such that $\omega + 1/4 - 1/9 = \pi^2(1+\epsilon)^2$. Since $\sin(\delta + \pi(1+\epsilon)t)$ vanishes twice in $(0,1)$ when $\delta > 0$ is small, a Sturm comparison with $\theta'' + \pi^2(1+\epsilon)^2\theta = 0$ shows that every solution of the latter of the displayed equations has a zero in $(0,1)$. Therefore every solution of the former does, also. □

3. Counting zeroes in the unit disk

We now prove Theorems 1 and 2.

Lemma 15. *If p is a holomorphic function in a disk $|z - z_0| < r$ and $|p| \le C$, then nontrivial solutions of $u'' + pu = 0$ with $u(z_0) = 0$ have at most $1 + r\sqrt{C}/\log 2$ zeroes in the disk $|z - z_0| \le r/2$.*

Proof. One may assume that $C > 0$, the assertion being clear otherwise, and that the disk is the unit disk, for the change of variable $z = z_0 + r\zeta$ transforms the general case to an equation $v'' + Pv = 0$ in $\mathbb{D}$ in which $|P| \le r^2 C$.

Under those assumptions, let u be a nontrivial solution of $u'' + pu = 0$ that vanishes at the origin. If $w(r) = \sqrt{C}\,|u(re^{i\theta})| + |u'(re^{i\theta}) - u'(0)|$, where θ is fixed, then, for all $r \in [0,1)$,

$$\begin{aligned} w(r) &= \sqrt{C}\left|\int_0^r u'(te^{i\theta})e^{i\theta}\,dt\right| + \left|-\int_0^r p(te^{i\theta})u(te^{i\theta})e^{i\theta}\,dt\right| \\ &\le \int_0^r \Big(\sqrt{C}\,|u'(te^{i\theta})| + C\big|u(te^{i\theta})\big|\Big)dt \le \int_0^r \sqrt{C}\Big(|u'(0)| + w(t)\Big)dt. \end{aligned}$$

By Gronwall's inequality [9], it follows that $w(r) \le |u'(0)|(e^{\sqrt{C}\,r} - 1)$. Therefore

$$\big|u(re^{i\theta})\big| \le |u'(0)| \cdot \frac{e^{\sqrt{C}\,r} - 1}{\sqrt{C}} \le |u'(0)| \cdot re^{\sqrt{C}\,r} < |u'(0)|e^{\sqrt{C}}, \qquad r \in [0,1).$$

Let $z_1, \dots, z_{N_r}$ be the zeroes of u in an annulus $0 < |z| \le r$, where $r \in (\frac{1}{2}, 1)$. By Jensen's formula and the bound on $|u|$,

$$\log r + \log\bigl|u'(0)\bigr| + \sum_{j=1}^{N_r} \log\left(\frac{r}{|z_j|}\right) = \frac{1}{2\pi}\int_0^{2\pi} \log\bigl|u(re^{i\theta})\bigr|d\theta \le \log\bigl|u'(0)\bigr| + \sqrt{C}.$$

It follows that $N_{1/2}\log(2r) \le \sqrt{C} - \log r$, and letting $r \to 1$ yields $N_{1/2} \le \sqrt{C}/\log 2$.

□

Proof of Theorem 1: The hypotheses of Theorem 1 provide a nontrivial solution of an equation $u'' + pu = 0$ in $\mathbb{D}$ and a number $R \in [0,1)$ such that $|p(z)| \le 1/(1-|z|^2)^2$ whenever $R \le |z| < 1$. Again, let $M_p(r) = \max\{|p(z)| : |z| = r\}$.

Consider a region $W = \{z : 1-2a \le |z| < 1-a\}$, where $a \in (0,1)$; this is an annulus if $a < \frac{1}{2}$ and a disk if $a \ge \frac{1}{2}$. Among the zeroes of u in W, let $b_1, \dots, b_m$ be a maximal collection with the property that $|b_i - b_j| \ge a/4$ when $i \ne j$. The open disks $D(b_j, a/8)$ being pairwise disjoint and contained in the $a/8$-neighborhood of W, one sees from a computation of areas that $m \le 160/a$. Here $|p| \le M_p(1-a/2)$ throughout $D(b_j, a/2)$, and by construction the union of the disks $D(b_j, a/4)$ contains all the zeroes of u in W. By Lemma 15, it follows that

$$\#\{z \in W : u(z) = 0\} \le \frac{160}{a}\left(1 + \frac{(a/2)\sqrt{M_p(1-a/2)}}{\log 2}\right).$$

Let $\alpha = (1-R)/2$. Partitioning the disk $|z| < (1+R)/2$ into such regions corresponding to values $a = \alpha, 2\alpha, \dots, 2^K\alpha$ and summing yields

$$\#\left\{z : u(z) = 0,\ |z| < \frac{1+R}{2}\right\} < \frac{320}{\alpha} + \frac{80}{\log 2}\sum_{k=0}^{K}\sqrt{M_p\bigl(1 - 2^{k-1}\alpha\bigr)}.$$

The sum here, written in the form $2\sum_{k=0}^{K} 2^{k-2}\alpha \cdot \{M_p(1-2^{k-1}\alpha)\}^{1/2}/(2^{k-1}\alpha)$, is twice a left Riemann sum for the integral of $M_p(r)^{1/2}/(1-r)$ from the point $1 - 2^{K-1}\alpha > 0$ to $1 - \alpha/4$. Because the integrand is nondecreasing, the Riemann sum is less than or equal to the integral; furthermore, the integrand is bounded by $1/(1-r)^2$ when $r \ge R = 1 - 2\alpha$. It follows that

$$\begin{aligned}\#\left\{z : u(z) = 0,\ |z| < \frac{1+R}{2}\right\} &\le \frac{320}{\alpha} + \frac{160}{\log 2}\int_0^{1-\alpha/4}\frac{\sqrt{M_p(r)}}{1-r}\,dr \\ &\le \frac{640}{1-R} + \frac{160}{\log 2}\left(\int_0^R \frac{\sqrt{M_p(r)}}{1-r}\,dr + \frac{7}{1-R}\right).\end{aligned}$$

As shown on p. 27 of [2], the remaining annulus $(1+R)/2 \le |z| < 1$ can be covered by at most $5/(1-R)$ hyperbolic half-planes—intersections of the unit disk with open disks whose boundaries are orthogonal to the unit circle—that are in turn contained in $R \le |z| < 1$. Each such set includes at most one zero of u, for any two points in it are contained in a curve Γ that satisfies the hypotheses of Theorem 12. Adding $5/(1-R)$ to the bound above then establishes Theorem 1 with $A = 645 + 1120/\log 2$ and $B = 160/\log 2$. □

Proof of Theorem 2: Theorem 2 concerns the maximum $N_\alpha(C)$ of the oscillation numbers among the equations $u'' + pu = 0$ in $\mathbb{D}$ in which

$$\bigl|p(z)\bigr| \le \frac{C}{(1-|z|^2)^{2\alpha}}\,, \qquad z \in \mathbb{D}, \tag{6}$$

where $\alpha \in (0,1)$ is fixed. The theorem asserts that there are positive numbers k_α, K_α, and C_α such that $k_\alpha C^{1/(2-2\alpha)} \le N_\alpha(C) \le K_\alpha C^{1/(2-2\alpha)}$ when $C \ge A_\alpha$.

Let A and B be as in Theorem 1. If $C \ge 1$, and if $R \in [0,1)$ is the solution of $C(1-R^2)^{-2\alpha} = (1-R^2)^{-2}$, then $(1-R)^{-1} < 2(1-R^2)^{-1} = 2C^{1/(2-2\alpha)}$, and

$$\int_0^R \frac{\sqrt{M_p(r)}}{1-r}\,dr \le \int_0^R \frac{\sqrt{C}}{(1-r)^{1+\alpha}}\,dr < \frac{\sqrt{C}}{\alpha}\cdot\bigl(2C^{1/(2-2\alpha)}\bigr)^\alpha < (2/\alpha)C^{1/(2-2\alpha)}$$

when p satisfies (6). By Theorem 1, it follows that $N_\alpha(C) \le (2A+2B/\alpha)C^{1/(2-2\alpha)}$.

To establish a lower bound of the same order, let $p_k(z) = \omega(k/2+1)^2 z^k$, where k is a positive integer and ω is as in Lemma 14. By that lemma, the oscillation number of the equation $u'' + p_k u = 0$ is at least $k+3$, and one easily sees that $|p_k(z)| \le C_k/(1-|z|^2)^{2\alpha}$, where $C_k = 4\omega(k+2)^{2-2\alpha}$. Therefore

$$N_\alpha(C_k) > k+2 = \eta \cdot C_k^{1/(2-2\alpha)}, \qquad \eta = (4\omega)^{-1/(2-2\alpha)}.$$

Because $C_k^{1/(2-2\alpha)} > \frac{1}{2}C_{k+1}^{1/(2-2\alpha)}$, it follows that

$$N_\alpha(C) \ge N_\alpha(C_k) > (\eta/2)C_{k+1}^{1/(2-2\alpha)} > (\eta/2)C^{1/(2-2\alpha)}, \qquad C \in [C_k, C_{k+1}).$$

This bound applies whenever $C \ge C_1$, and the proof is complete. □

Theorem 1 also gives the bound $N_0(C) = O(\sqrt{C}\,\log C)$ for equations $u''+pu=0$ in which $|p| \le C$. We have no evidence, however, that $N_0(C)$ is larger than $O(\sqrt{C}\,)$.

One might conjecture, based on constant-coefficient equations, that it is asymptotic to $2\sqrt{C}/\pi$ as $C \to \infty$, but it is actually larger than that, for if p_k is as above then

$$\liminf_{C\to\infty} \frac{N_0(C)}{\sqrt{C}} \geq \liminf_{k\to\infty} \frac{k+3}{(\sup_{\mathbb{D}} |p_k|)^{1/2}} = \liminf_{k\to\infty} \frac{k+3}{\sqrt{\omega}\,(k/2+1)} = \frac{2}{\sqrt{\omega}} > \frac{2}{\pi}\,.$$

A related open question is whether the oscillation numbers $N_p(C)$ of $u'' + Cpu = 0$ are $O(\sqrt{C})$ for every bounded holomorphic function p in $\mathbb{D}$.

4. Conditions that allow infinite oscillation in the disk

Theorem 5 asserts that, if $\beta : [0,1) \to (0,\infty)$ is continuous and

$$\lim_{r\to 1} \frac{\beta(r)-1}{1-r} = \infty, \tag{7}$$

then there is a holomorphic function p in $\mathbb{D}$ such that $[\![p]\!](z) \leq \beta(|z|)$ for all $z \in \mathbb{D}$ and some nontrivial solution of $u'' + pu = 0$ has infinitely many zeroes. Because quotients of linearly independent solutions of that equation are the solutions of $S(f) = 2p$, Theorem 5 is equivalent to the following result, which we prove here:

Theorem 16. *If $\beta : [0,1) \to (0,\infty)$ is continuous and (7) holds, then there is a locally injective, meromorphic function G in $\mathbb{D}$ that satisfies $\frac{1}{2}[\![S(G)]\!](z) \leq \beta(|z|)$ for all $z \in \mathbb{D}$ and attains some value infinitely many times.*

The construction is based on mappings F_μ and $F_{\mu\alpha}$ for $\mu, \alpha \in [0,1)$, the former defined as solutions of nonlinear initial-value problems and the latter as conjugates of those by Möbius transformations T_α. Using them, we prove:

Lemma 17. *Let $\gamma : [0,1) \to (0,\infty)$ be continuous with $(\gamma(r)-1)/(1-r) \to \infty$ as $r \to 1$. If $C > 0$ and $\zeta \in \partial\mathbb{D}$, then for all sufficiently small $\epsilon > 0$ there is a locally injective, holomorphic function f in a neighborhood of* $\operatorname{clos}(\mathbb{D})$ *such that*

(i) *$f(\mathbb{D}) \subseteq \mathbb{D}$,*

(ii) *$\frac{1}{2}[\![S(f)]\!](z) \leq \gamma(|z|) - C\epsilon^2$ for all $z \in \mathbb{D}$,*

(iii) *$|f(z) - z| < \epsilon$ for all $z \in \mathbb{D}$, and*

(iv) *$f(z) = f(\zeta')$ for some $z \in \mathbb{D}$ with $|z - \zeta| < 2\epsilon$ and some $\zeta' \in \partial\mathbb{D}$.*

The mappings G that establish Theorem 16 will be limits of the restrictions, to $\mathbb{D}$, of composites $f_1 \circ \cdots \circ f_n$ of such functions f; in particular, they will be holomorphic. We prove the theorem as follows:

Step 1. Deduce Theorem 16 from Lemma 17.

Step 2. Define the mappings F_μ and $F_{\mu\alpha}$ and establish their properties.

Step 3. Use those properties to prove Lemma 17.

Step 1 is logically last, and we do not use it in steps 2 and 3. Step 2 culminates in Lemma 21, a summary of key properties of the mappings $F_{\mu\alpha}$. Those reflect properties of F_μ and T_α that are for the most part quite evident, but the fact that $F_\mu \to F_0$ uniformly in $\mathbb{D}$ as $\mu \to 0$, while expected, emerges only after some preliminary lemmas. For that reason, step 2 occupies much of the argument.

Step 1. Assume that Lemma 17 is valid. Given β as in Theorem 16, we construct the mapping G that the theorem promises as a limit of mappings $G_n = f_1 \circ \cdots \circ f_n|_{\mathbb{D}}$, where f_k satisfies the conditions in the lemma for data γ_k, C_k, ζ_k and ϵ_k. Because $f_1 \circ \cdots \circ f_n$ is holomorphic and locally injective in a neighborhood of $\mathrm{clos}(\mathbb{D})$, each mapping G_n will be uniformly continuous. In view of (iii) in the lemma, one can therefore make $\sup |G_{n+1} - G_n|$ as small as desired by making ϵ_{n+1} small and so assure that the sequence $\{G_n\}$ converges uniformly to a function G, with $\sup |G - G_1|$ small enough to guarantee that G is not constant. Further requirements will be imposed on the numbers ϵ_n, but this one is implicit in the arguments below.

Let $0 < \beta_1 < \beta_2 < \cdots < \beta$ be continuous functions in $[0, 1)$ that satisfy (7), such as $\beta_n(t) = \beta(t) - (\inf \beta)(1 - t)/(n + 1)$. The construction will be such that

(a) $\frac{1}{2}[\![S(G_n)]\!](z) \le \beta_n(|z|)$ for all $z \in \mathbb{D}$,

(b) the continuous extension $f_1 \circ \cdots \circ f_n$ of G_n to $\mathrm{clos}(\mathbb{D})$ maps distinct points $z_{n1}, \ldots, z_{nn} \in \mathbb{D}$ and $\zeta_n \in \partial\mathbb{D}$ to a common image, and

(c) there are pairwise-disjoint closed disks $D_{nk} \subseteq \mathbb{D}$ centered at the points z_{nk} such that $D_{nk} \subseteq D_{n-1,k}$ and $\mathrm{diam}(D_{nk}) \le \frac{1}{2}\,\mathrm{diam}(D_{n-1,k})$ when $k \le n-1$.

The limit G then fulfills the conditions of Theorem 16. Indeed, by (c) the sequences $z_{kk}, z_{k+1,k}, \ldots$ converge in $\mathbb{D}$ to distinct limits, and by (b) and the uniform convergence $G_n \to G$ those limits map to the same image under G. Since G is not constant and the functions G_n are locally injective, Hurwitz's theorem implies that G is locally injective. Finally, $\frac{1}{2}[\![S(G)]\!](z) = \lim \frac{1}{2}[\![S(G_n)]\!](z)| \leq \beta(|z|)$ for all $z \in \mathbb{D}$.

To achieve (a)–(c) when $n = 1$, one can let $G_1 = f|_{\mathbb{D}}$, where f satisfies the conditions of Lemma 17 for the function $\gamma = \beta_1$ and some C, ζ, and ϵ. Condition (iv) there provides points $z \in \mathbb{D}$ and $\zeta' \in \partial\mathbb{D}$ such that $f(z) = f(\zeta')$, and one can let $z_{11} = z$ and $\zeta_1 = \zeta'$ and let D_{11} be the disk $|w - z_{11}| \leq (1 - |z_{11}|)/2$.

Proceeding by induction, suppose that (a)–(c) hold for some $n \geq 1$. Because G_n extends to a locally injective function in a neighborhood of $\mathrm{clos}(\mathbb{D})$, its Schwarzian derivative is bounded: say, $|S(G_n)| \leq M$. Let $\alpha = (\beta_n + \beta_{n+1})/2$. Increasing M if necessary, we assume that $3M/2 > \alpha(0)$. Since $\alpha(r) > (3M/2)(1 - r^2)$ for all r near one and the reverse inequality holds when $r = 0$, there is a largest number $R \in (0, 1)$ at which equality holds. The function

$$\gamma(r) = \begin{cases} \beta_{n+1}(r) - \alpha(r) & \text{if } r \in [0, R] \\ \beta_{n+1}(r) - (3M/2)(1 - r^2) & \text{if } r \in [R, 1) \end{cases}$$

then satisfies the hypothesis in Lemma 17: It is continuous and positive, and the ratio $(\gamma(r) - 1)/(1 - r)$ approaches infinity as $r \to 1$ since β_{n+1} has that property. We let $G_{n+1} = G_n \circ f|_{\mathbb{D}}$, where f satisfies the conclusions of the lemma for this function γ, the value $C = 2M$, the point $\zeta = \zeta_n$ that the inductive hypothesis (b) provides, and some $\epsilon \in (0, \frac{1}{2}]$. The arguments below show that if ϵ is sufficiently small then conditions (a)–(c) hold at the next level $n' = n + 1$.

Since $\beta_n < \alpha$, there exists $\eta > 0$ such that if $r \in [0, R]$, $s \in [0, 1)$, and $|s - r| < \eta$ then $\beta_n(s) < \alpha(r)$. We first show that if $\epsilon \leq \eta$ then $\frac{1}{2}[\![S(G_{n+1})]\!](z) \leq \beta_{n+1}(|z|)$, so that G_{n+1} satisfies (a). By the chain rule (3) for the Schwarzian derivative,

$$SG_{n+1}(z) = Sf(z) + f'(z)^2 \cdot SG_n\big(f(z)\big), \qquad z \in \mathbb{D}. \tag{8}$$

Suppose that $|z| \in [0, R]$. By properties (i) and (iii) in Lemma 17, $|f(z)|$ is less than one and within ϵ units of $|z|$, and since $\epsilon \leq \eta$ it follows that $\beta_n(|f(z)|) < \alpha(|z|)$.

Using the Schwarz lemma and the inductive hypothesis (a), one then has

$$\left|f'(z)^2 \cdot SG_n\big(f(z)\big)\right| \le \left(\frac{1-|f(z)|^2}{1-|z|^2}\right)^2 \cdot \frac{2\beta_n\big(|f(z)|\big)}{(1-|f(z)|^2)^2} \le \frac{2\alpha\big(|z|\big)}{(1-|z|^2)^2}\,,$$

and by (8) and property (ii) in Lemma 17 it follows that

$$\tfrac{1}{2}[\![S(G_{n+1})]\!](z) \le \Big(\beta_{n+1}\big(|z|\big) - \alpha\big(|z|\big) - 2M\epsilon^2\Big) + \alpha\big(|z|\big) < \beta_{n+1}\big(|z|\big).$$

Suppose, then, that $|z| \in [R, 1)$. Again by the Schwarz lemma,

$$\begin{aligned}\left|f'(z)^2 \cdot SG_n\big(f(z)\big)\right| &\le \left(\frac{1-|f(z)|^2}{1-|z|^2}\right)^2 \cdot M \le \frac{M\big(1-|z|^2+2\epsilon\big)^2}{(1-|z|^2)^2} \\ &\le \frac{M\big\{3(1-|z|^2)+4\epsilon^2\big\}}{(1-|z|^2)^2}\,;\end{aligned}$$

the first inequality holds since $|S(G_n)| \le M$, the second since $|z| - \epsilon < |f(z)| < 1$ as noted above, and the third since $\epsilon \le \frac{1}{2}$. For such z, equation (8) and property (ii) in Lemma 17 then imply that $\frac{1}{2}[\![S(G_{n+1})]\!](z)$ is no greater than

$$\Big(\beta_{n+1}(|z|) - (3M/2)\big(1-|z|^2\big) - 2M\epsilon^2\Big) + \frac{M}{2}\Big(3\big(1-|z|^2\big) + 4\epsilon^2\Big) = \beta_{n+1}\big(|z|\big).$$

Thus condition (a) persists to the next inductive step if $\epsilon \le \eta$.

The inductive hypothesis (b) provides points $z_{n1}, \ldots, z_{nn} \in \mathbb{D}$ and $\zeta_n \in \partial\mathbb{D}$ that map to a common image w under $f_1 \circ \cdots \circ f_n$, and (c) provides pairwise-disjoint closed disks $D_{nk} \subseteq \mathbb{D}$ with center z_{nk}. Since G_n is not constant, a standard use of Rouché's theorem produces a closed disk D centered at w and a number $\delta > 0$ such that, if $H : \mathbb{D} \to \mathbb{C}$ is holomorphic and $|H - G_n| < \delta$, then every image $H(\mathrm{int}(D_{nk}))$ contains D. By the uniform continuity of G_n and property (iii) in Lemma 17, the function $H = G_{n+1}$ satisfies that condition when ϵ is sufficiently small. A further restriction $\epsilon \le \epsilon_*$ assures that the region $R = \{z \in \mathbb{D} : |z - \zeta_n| < 3\epsilon\}$ is contained in $G_n^{-1}(D)$ and disjoint from $D_{n1}, \ldots, D_{nn}$. Suppose that ϵ satisfies all these conditions. By (iv) in the lemma, f maps points $z = z_{n+1,n+1} \in \mathbb{D}$ with $|z - \zeta_n| < 2\epsilon$ and $\zeta' = \zeta_{n+1} \in \partial\mathbb{D}$ to a common image, and by (i) and (iii) in the lemma that image is in R. Therefore G_{n+1} maps $z_{n+1,n+1}$ and ζ_n to a common image $w' \in D$, and it also maps a point $z_{n+1,k}$ in each set $\mathrm{int}(D_{nk})$ to w'. Small closed disks centered at these new points then perpetuate conditions (b) and (c) to the next inductive step, and the proof of Theorem 16 is complete.

Step 2. Let D be the strip $|\mathrm{Re}(z)| < 1$, and for $\mu \in [0,1)$ let F_μ be the solution of

$$S(F) = 2p_\mu, \qquad p_\mu(z) = \frac{1-\mu^2+2\mu(1-z^2)}{(1-z^2)^2}\,,$$

in D with $(F, F', F'')(i) = (i, 1, 0)$; thus $F_\mu = i + Y_\mu/X_\mu$, where X_μ and Y_μ are the solutions of $u'' + p_\mu u = 0$ in D with $(X_\mu, X'_\mu)(i) = (1,0)$ and $(Y_\mu, Y'_\mu)(i) = (0,1)$.

The initial mapping F_0 is given by

$$F_0(z) = \frac{(4-\pi)i + 2L(z)}{4+\pi+2iL(z)}\,, \qquad L(z) = \log\left(\frac{1+z}{1-z}\right). \tag{9}$$

Here L maps $\mathbb{D}$ conformally onto the strip $|\mathrm{Im}(w)| < \pi/2$, with $L(z)$ tending to ∞ as $z \to \pm 1$, and the Möbius transformation $w \mapsto ((4-\pi)i + 2w)/(4+\pi+2iw)$ takes that strip into $\mathbb{D}$, mapping the upper boundary to the unit circle and ∞ to $-i$. Thus F_0 maps neighborhoods of ± 1 in $\mathbb{D}$ to two cuspidal regions that meet at the point $-i$. As μ increases, the images of such neighborhoods under F_μ begin to push into each other; indeed, the proof of (a) in Lemma 21 below shows that F_μ fails to be injective in $(-1,1)$ when μ is small and positive.

We move that non-injectivity into a neighborhood of $-i$ by means of secondary deformations $F_{\mu\alpha}$ for $\alpha \in [0,1)$, defined by

$$F_{\mu\alpha} = T_{-\alpha} \circ F_\mu \circ T_\alpha, \qquad T_\alpha(z) = \frac{z+i\alpha}{1-i\alpha z}\,.$$

Lemma 21 below gives the key properties of these mappings; assertions of nearness or smallness in the following overview appear there as bounds in terms of $1-\alpha$ that hold for all $\alpha \in [0,1)$ when μ is sufficiently small. The first property is that $F_{\mu\alpha}$ maps two points in $\mathbb{D}$ that are near $-i$ to a common image. The second and third assert that $\frac{1}{2}[\![S(F_{\mu\alpha})]\!](z)$ is small when $|z|$ is in a substantial interval $[0, \sigma_\alpha] \subseteq [0,1)$ and that $\frac{1}{2}[\![S(F_{\mu\alpha})]\!](z) \le 1 - \mu^2 + 8\mu(1-|z|)/(1-\alpha)$ for all z. Those derive from the geometry of T_α and the identity

$$\tfrac{1}{2}[\![S(F_{\mu\alpha})]\!](z) = \tfrac{1}{2}[\![S(F_\mu \circ T_\alpha)]\!](z) = \tfrac{1}{2}[\![S(F_\mu)]\!]\big(T_\alpha(z)\big) = [\![p_\mu]\!]\big(T_\alpha(z)\big) \tag{10}$$

for $z \in \mathbb{D}$, as in (5). The second property, for example, reflects the fact that $T_\alpha(z)$ is near i when $|z| \le \sigma_\alpha$, for the regularity of p_μ at i results in $[\![p_\mu]\!]$ being small at nearby points in $\mathbb{D}$. A final property is that $\sup_{z\in\mathbb{D}} |F_{\mu\alpha}(z) - z|$ is small when α is near one. That is largely a consequence of bounds $|F_\mu(w) - w| \le C|w-i|^3$

for w near i and $\mu \in [0, \frac{1}{2}]$: If $z \in \mathbb{D}$ and α is near one, then $T_\alpha(z)$ is near i, the mapping F_μ moves that point very little, and applying $T_{-\alpha}$ returns a point near z. This idea, however, does not yield uniform bounds for $|F_{\mu\alpha}(z) - z|$ when μ and α are fixed. For that, one needs control of F_μ near ± 1, and Lemma 20 provides the control by showing that $F_\mu \to F_0$ uniformly in $\mathbb{D}$ as $\mu \to 0$.

To carry all this out, we compare F_μ with the solution G_μ of

$$S(G) = 2q_\mu, \qquad q_\mu(z) = \frac{1-\mu^2}{(1-z^2)^2}\,,$$

in D with $(G, G', G'')(i) = (i, 1, 0)$. Here $G_0 = F_0$, and if U_μ and V_μ are the solutions of $u'' + q_\mu u = 0$ with $(U_\mu, U'_\mu)(i) = (1, 0)$ and $(V_\mu, V'_\mu)(i) = (0, 1)$ then $G_\mu = i + V_\mu/U_\mu$. Those solutions are given by

$$\begin{aligned} U_\mu(z) &= (1-z^2)^{1/2}\big\{a(\mu)\cdot\cosh\big(\mu L(z)/2\big) + b(\mu)\cdot\mu^{-1}\sinh\big(\mu L(z)/2\big)\big\} \\ V_\mu(z) &= (1-z^2)^{1/2}\big\{c(\mu)\cdot\cosh\big(\mu L(z)/2\big) + d(\mu)\cdot\mu^{-1}\sinh\big(\mu L(z)/2\big)\big\} \end{aligned} \tag{11}$$

for some $a(\mu), b(\mu), c(\mu), d(\mu) \in \mathbb{C}$, where $\mu^{-1}\sinh(\mu w)$ is interpreted as w if $\mu = 0$ and, again, $L(z) = \log((1+z)/(1-z))$. The functions a, b, c, and d are of class C^∞, for the conditions $(U_\mu, U'_\mu)(i) = (1, 0)$ and $(V_\mu, V'_\mu)(i) = (0, 1)$ define nonsingular linear systems in which the coefficients are C^∞ functions of μ.

Lemma 18. *$G_\mu(\mathbb{D}) \subseteq \mathbb{C}$ when $\mu > 0$ is small, and $\sup_{\mathbb{D}} |G_\mu - G_0|$ is $O(\mu)$ as $\mu \to 0$.*

Proof. The mapping G_μ is the composite $M_\mu \circ H_\mu \circ T$, where

$$M_\mu(\zeta) = i + \frac{c(\mu)\zeta + d(\mu)}{a(\mu)\zeta + b(\mu)}\,, \qquad H_\mu(w) = \frac{\mu(w^\mu + 1)}{w^\mu - 1}\,, \qquad T(z) = \frac{1+z}{1-z}\,,$$

with $H_0(w) = 2/\log w$. Since T maps $\mathbb{D}$ onto the right half-plane $\mathbb{H}$, the assertion in the lemma is equivalent to the statement that $\sup_{\mathbb{H}} |M_\mu \circ H_\mu - M_0 \circ H_0|$ is $O(\mu)$. Note that, by (9), the set $(M_0 \circ H_0)(\mathbb{H}) = G_0(\mathbb{D}) = F_0(\mathbb{D})$ is contained in $\mathbb{D}$.

We first show that $\sup_{\mathbb{H}} |M_0 \circ H_\mu - M_0 \circ H_0|$ is $O(\mu)$. The partial derivative $\partial_\mu(H_\mu(w))$ equals $f(w^\mu)$, where $f(u) = (u+1)/(u-1) - 2u\log u/(u-1)^2$. The singularity of f at 1 being removable, a bound $|f| \le c$ holds in $\mathbb{H}$, and since $w^\mu \in \mathbb{H}$ when $w \in \mathbb{H}$ and $\mu \in [0, 1)$ one sees by integrating that $|H_\mu(w) - H_0(w)| \le c\mu$ for all such μ and w with $w \neq 1$. Here $H_\mu(1) \equiv \infty$. Because $(M_0 \circ H_0)(\mathbb{H}) \subseteq \mathbb{D}$, it follows that the point $z_0 = M_0^{-1}(\infty)$ is in the finite plane and that $H_0(\mathbb{H})$ omits some disk

$|z - z_0| < r$. But then $H_\mu(\mathbb{H})$ omits the disk $|z - z_0| < r/2$ when $\mu \le r/(2c)$. Since M_0' is bounded outside such a disk, a similar integration of $\partial_\mu((M_0 \circ H_\mu)(w))$ establishes a bound $\sup_{\mathbb{H}} |M_0 \circ H_\mu - M_0 \circ H_0| \le C\mu$ for all μ in some interval $[0, s]$.

Let D be the disk $|z| \le 1 + Cs$, so that $(M_0 \circ H_\mu)(\mathbb{H}) \subseteq D$ when $\mu \in [0, s]$. Since the mapping $(\mu, z) \mapsto (M_\mu \circ M_0^{-1})(z)$ is C^∞ in a neighborhood of $\{0\} \times D$, it is Lipschitz with respect to μ in some set $[0, s'] \times D$. Hence there exists K such that

$$\sup_{\mathbb{H}} \big|(M_\mu \circ M_0^{-1}) \circ (M_0 \circ H_\mu) - M_0 \circ H_\mu\big| \le K\mu, \qquad 0 \le \mu \le \min\{s, s'\},$$

and for such μ one has $\sup_{\mathbb{H}} |M_\mu \circ H_\mu - M_0 \circ H_0| \le (C + K)\mu$. □

Our real objective is to show that $\sup_{\mathbb{D}} |F_\mu - F_0|$ is $O(\mu)$ as $\mu \to 0$. That is a consequence of Lemma 18 and the following:

Lemma 19. *If $m_\mu = |U_\mu| + |V_\mu|$, then there exists C such that*

$$\sup_{z \in \mathbb{D}} \frac{\big|X_\mu(z) - U_\mu(z)\big|}{m_\mu(z)} \le C\mu, \quad \sup_{z \in \mathbb{D}} \frac{\big|Y_\mu(z) - V_\mu(z)\big|}{m_\mu(z)} \le C\mu, \qquad \mu \in (0, \tfrac{1}{2}].$$

Proof. We first claim that a bound $m_\mu(z) \le k|1 - z^2|^{1/8}$ holds for all $z \in \mathbb{D}$ and $\mu \in (0, \frac{1}{2}]$. One easily sees that the function $L(z) = \log((1 + z)/(1 - z))$ in (11) satisfies $|L(z)| < \pi/2 + \log 4 - \log|1 - z^2|$ when $z \in \mathbb{D}$. Because $|\cosh w| \le e^{|w|}$ and $|\sinh w| \le |w|e^{|w|}$ for all $w \in \mathbb{C}$, it then follows that both $(1 - z^2)^{1/2}\cosh(\mu L(z)/2)$ and $(1 - z^2)^{1/2}\mu^{-1}\sinh(\mu L(z)/2)$ are less than or equal to

$$|1 - z^2|^{1/2}\left(\frac{4e^{\pi/2}}{|1 - z^2|}\right)^{\mu/2} \cdot \frac{1}{2}\left(\frac{\pi}{2} + \log 4 - \log\big|1 - z^2\big|\right).$$

This expression is bounded by a constant times $|1 - z^2|^{1/2 - 1/4 - 1/8}$ for $z \in \mathbb{D}$ and $\mu \in (0, \frac{1}{2}]$, and the claim follows since a, b, c, and d are bounded in $(0, \frac{1}{2}]$.

The function $h = X_\mu - U_\mu$ satisfies

$$h'' + \frac{1 - \mu^2}{(1 - z^2)^2}\, h = -\frac{2\mu}{1 - z^2}\big(U_\mu + h\big), \qquad h(i) = h'(i) = 0.$$

We apply the variation-of-parameters formula, as in [4]. Given $z \in \mathbb{D}$, let $\pi(s)$ be the arc-length parametrization, by $[-1, |z|]$, of the segment from i to the origin followed by that from the origin to z. By variation of parameters,

$$h(z) = \int_\pi \Gamma_\mu(z, \zeta) \cdot \left(\frac{-2\mu}{1 - \zeta^2}\right)\big(U_\mu(\zeta) + h(\zeta)\big)d\zeta,$$

where $\Gamma_\mu(z,\zeta) = U_\mu(\zeta)V_\mu(z) - V_\mu(\zeta)U_\mu(z)$. Using the bound $m_\mu(\zeta) \le k|1-\zeta^2|^{1/8}$ and the evident bounds $|\Gamma_\mu(z,\zeta)| \le m_\mu(z)m_\mu(\zeta)$ and $|U_\mu|/m_\mu \le 1$, one obtains

$$\begin{aligned}\frac{|h(z)|}{m_\mu(z)} &\le \frac{1}{m_\mu(z)}\int_\pi \frac{m_\mu(z)m_\mu(\zeta)\cdot 2\mu}{|1-\zeta^2|}\cdot m_\mu(\zeta)\Big(1+\frac{|h(\zeta)|}{m_\mu(\zeta)}\Big)|d\zeta| \\ &\le 2\mu\int_\pi \frac{k^2}{|1-\zeta^2|^{3/4}}\cdot\Big(1+\frac{|h(\zeta)|}{m_\mu(\zeta)}\Big)|d\zeta|.\end{aligned}$$

Since $|1-\zeta^2| \ge 1$ in the segment from i to the origin and $|1-\zeta^2| \ge 1-|\zeta|$ in that from the origin to z, the function $w(s) = |h(\pi(s))|/m_\mu(\pi(s))$ satisfies

$$w(t) \le 2\mu k^2\int_{-1}^t f(s)\big(1+w(s)\big)ds, \qquad f(s) = \begin{cases} 1 & \text{if } s\in[-1,0] \\ (1-s)^{-3/4} & \text{if } s\in[0,|z|]. \end{cases}$$

By Gronwall's inequality [9], w is no greater than the solution of the corresponding integral equation. Solving that equation, one concludes that

$$\frac{|h(z)|}{m_\mu(z)} \le -1+\exp\big\{10\mu k^2 - 8\mu k^2(1-|z|)^{1/4}\big\} < e^{10\mu k^2}-1 \le 10k^2e^{5k^2}\mu$$

for all $z\in\mathbb{D}$, $\mu\in[0,\frac{1}{2}]$. A similar proof yields the same bound for $|Y_\mu - V_\mu|/m_\mu$. □

Lemma 20. *$F_\mu(\mathbb{D}) \subseteq \mathbb{C}$ when $\mu > 0$ is small, and $\sup_{\mathbb{D}}|F_\mu - F_0|$ is $O(\mu)$ as $\mu \to 0$.*

Proof. In view of Lemma 18, it is enough to prove that $\sup_{\mathbb{D}}|F_\mu - G_\mu|$ is $O(\mu)$. Assume for the moment that there are positive numbers μ_* and δ_* such that $\inf_{\mathbb{D}}(|U_\mu|/m_\mu) \ge \delta_*$ when $\mu\in(0,\mu_*]$. The functions X_μ then enjoy a similar property, say, $\inf_{\mathbb{D}}(|X_\mu|/m_\mu) \ge \delta' > 0$ when $\mu\in(0,\mu']$, for $\sup_{\mathbb{D}}(|X_\mu - U_\mu|/m_\mu)$ is $O(\mu)$ by Lemma 19. From the same lemma and the evident bound $|V_\mu|/m_\mu \le 1$, one also sees that $\sup_{\mathbb{D}}(|Y_\mu|/m_\mu) \le 2$ when μ is small. If μ is small enough that all these conditions hold, then

$$\begin{aligned}\big|F_\mu - G_\mu\big| = \left|\frac{Y_\mu}{X_\mu} - \frac{V_\mu}{U_\mu}\right| &= \left|\frac{Y_\mu - V_\mu}{U_\mu} - \frac{(X_\mu - U_\mu)Y_\mu}{X_\mu U_\mu}\right| \\ &\le \frac{|Y_\mu - V_\mu|}{m_\mu}\cdot\frac{1}{\delta_*} + \frac{|X_\mu - U_\mu|}{m_\mu}\cdot\frac{2}{\delta_*\delta'}\end{aligned}$$

throughout $\mathbb{D}$, and by Lemma 19 it follows that $\sup_{\mathbb{D}}|F_\mu - G_\mu|$ is $O(\mu)$.

It remains to show that such numbers μ_* and δ_* exist. A first claim is that, for all $M > 0$, there exist $\mu_1^M \in (0,1)$ and punctured neighborhoods $\Omega_\pm^M$ of ± 1 in $\mathbb{C}$ such that the functions $\tau_\mu(z) = \mu^{-1}\tanh(\mu L(z)/2)$ with $L(z) = \log((1+z)/(1-z))$

satisfy

$$\big|\tau_\mu(z)\big| > M, \qquad z \in \mathbb{D} \cap (\Omega_+^M \cup \Omega_-^M), \quad \mu \in \big(0, \mu_1^M\big]. \tag{12}$$

Let $T(z) = (1+z)/(1-z)$. The set $\Omega_+^M = \{z \in \mathbb{C} : |T(z)| > e^{4M}\}$ is a punctured neighborhood of 1, and if $z \in \mathbb{D} \cap \Omega_+^M$ then

$$\big|\tau_\mu(z)\big| = \frac{1}{\mu}\left|\frac{T(z)^\mu - 1}{T(z)^\mu + 1}\right| \geq \frac{1}{\mu} \cdot \frac{1 - |T(z)|^{-\mu}}{1 + |T(z)|^{-\mu}} > \frac{1 - e^{-4M\mu}}{2\mu}.$$

The latter quotient approaches $2M$ as $\mu \to 0$, so it exceeds M for all μ in some interval $(0, \mu_1^M]$. The claim then holds with $\Omega_-^M = -\Omega_+^M$, for $\tau_\mu(-z) = -\tau_\mu(z)$.

One computes that $b(0) = i/\sqrt{2}$. Let μ_2 be such that b remains nonzero throughout $(0, \mu_2]$, and in that interval let $A = |a/b|$, $C = |c/b|$, and $D = |d/b|$. By (12),

$$\begin{aligned}\frac{|U_\mu(z)|}{m_\mu(z)} &= \frac{|a(\mu) + b(\mu)\tau_\mu(z)|}{|a(\mu) + b(\mu)\tau_\mu(z)| + |c(\mu) + d(\mu)\tau_\mu(z)|} \\ &\geq \frac{-A(\mu) + M}{A(\mu) + M + C(\mu) + MD(\mu)}\end{aligned}$$

whenever $z \in \mathbb{D} \cap (\Omega_+^M \cup \Omega_-^M)$ and $\mu \leq \min\{\mu_1^M, \mu_2\}$. Since A, C, and D are bounded in $(0, \mu_2]$, the choice $M = 1 + \sup_{(0,\mu_2]} A$ yields positive numbers δ_1 and $\mu_3 = \min\{\mu_1^M, \mu_2\}$ and punctured neighborhoods $\Omega_\pm = \Omega_\pm^M$ of ± 1 such that $|U_\mu(z)|/m_\mu(z) \geq \delta_1$ whenever $z \in \mathbb{D} \cap (\Omega_+ \cup \Omega_-)$ and $\mu \in (0, \mu_3]$. Finally, consider the set $S = \operatorname{clos}(\mathbb{D}) - (\Omega_+ \cup \Omega_-)$. The function U_0 does not attain the value zero there, for (9) shows that the mapping $F_0 = i + V_0/U_0$ takes S into $\mathbb{C}$. Since S is compact, continuity then assures a bound $|U_\mu(z)|/m_\mu(z) \geq \delta_2 > 0$ for all $z \in S$ and all μ in some interval $(0, \mu_4]$. In all, these arguments show that the numbers $\mu_* = \min\{\mu_3, \mu_4\}$ and $\delta_* = \min\{\delta_1, \delta_2\}$ have the required properties. ☐

For $\alpha \in [0, 1)$, let $\sigma_\alpha = 1 - (1-\alpha)^{1/2}$; thus $0 \leq \sigma_\alpha \leq \alpha < 1$.

Lemma 21. *There exist $\mu_0 > 0$ and K such that, for all $\mu \in (0, \mu_0]$ and $\alpha \in [0, 1)$,*

(a) *$F_{\mu\alpha}$ fails to be injective in $\{z \in \mathbb{D} : |z + i| < K(1-\alpha)\}$,*

(b) *$\frac{1}{2}[\![S(F_{\mu\alpha})]\!](z) \leq K(1-\alpha)$ if $|z| \leq \sigma_\alpha$,*

(c) *$\frac{1}{2}[\![S(F_{\mu\alpha})]\!](z) \leq 1 - \mu^2 + 8\mu\big(1 - |z|\big)/(1-\alpha)$ for all $z \in \mathbb{D}$, and*

(d) *$\big|F_{\mu\alpha}(z) - z\big| < K(1-\alpha)$ for all $z \in \mathbb{D}$; in particular, $F_{\mu\alpha}(z) \in \mathbb{C}$.*

Proof. We treat these conditions separately; the lemma holds for the minimum of the numbers μ_0 that arise and the maximum of the numbers K. Easily verified properties of the mappings T_α will be used without proof.

By Lemma 13, some nontrivial solution of of $u'' + p_\mu u = 0$ vanishes at least twice in $(-1,1)$ when $\mu > 0$ is sufficiently small. Choosing an independent solution v and writing F_μ as $(Au+Bv)/(Cu+Dv)$, one sees that, for such μ, the mapping F_μ attains the value B/D at least twice in $(-1,1)$. Condition (a) then holds with $K = \sqrt{2}$, for $T_\alpha^{-1}(-1,1)$ lies within $\sqrt{2}\,(1-\alpha)$ units of $-i$.

For (b) and (c), we recall from (10) that $\frac{1}{2}[\![S(F_{\mu\alpha})]\!](z) = [\![p_\mu]\!](T_\alpha(z))$ when $z \in \mathbb{D}$. The asserted bounds derive from that identity and the fact that

$$\big|T_\alpha(z)\big| \geq \Big|T_\alpha\big(-i|z|\big)\Big| = \frac{\big|\alpha - |z|\,\big|}{1-\alpha|z|}\,, \qquad z \in \mathbb{D}, \quad \alpha \in [0,1). \tag{13}$$

Let M be the maximum of $|p_\mu(w)|$ for $\mu \in [0, \frac{1}{2}]$ and $w \in \operatorname{clos}(\mathbb{D})$ with $|w - i| \leq 1$. If $\alpha \in [0,1)$ and $|z| \leq \alpha$, then $|T_\alpha(z) - i| \leq 1$, and hence

$$\begin{aligned}\tfrac{1}{2}[\![S(F_{\mu\alpha})]\!](z) &= \big|p_\mu\big(T_\alpha(z)\big)\big| \cdot \big(1 - |T_\alpha(z)|^2\big)^2 \leq M\left\{1 - \left(\frac{\alpha - |z|}{1-\alpha|z|}\right)^2\right\}^2 \\ &= \frac{M(1-\alpha^2)^2(1-|z|^2)^2}{(1-\alpha|z|)^4} < \frac{16M(1-\alpha)^2}{(1-|z|)^2}\end{aligned}$$

for all $\mu \in [0, \frac{1}{2}]$. The latter bound is less than or equal to $16M(1-\alpha)$ when $|z| \leq \sigma_\alpha$, and (b) follows with $\mu_0 = \frac{1}{2}$ and $K = 16M$. For (c), note that

$$[\![p_\mu]\!](w) \leq [\![p_\mu]\!]\big(|w|\big) = 1 - \mu^2 + 2\mu\big(1 - |w|^2\big), \qquad w \in \mathbb{D}, \quad \mu \in [0,1),$$

for the Maclaurin series for p_μ has nonnegative coefficients. By (13), it follows that

$$\begin{aligned}\tfrac{1}{2}[\![S(F_{\mu\alpha})]\!](z) &\leq 1 - \mu^2 + 4\mu\big(1 - |T_\alpha(z)|\big) \leq 1 - \mu^2 + 4\mu\left(1 - \frac{|z| - \alpha}{1-\alpha|z|}\right) \\ &\leq 1 - \mu^2 + \frac{8\mu(1-|z|)}{1-\alpha}\,, \qquad z \in \mathbb{D}, \quad \mu \in [0,1), \quad \alpha \in [0,1).\end{aligned}$$

Therefore (c) holds regardless of μ_0.

It remains to establish (d). Let $U(z) = (i+z)/(i-z)$. This transformation maps $\mathbb{D}$ onto the right half-plane $\mathbb{H}$, and one computes that

$$\big|U(z) - U(z')\big| = \frac{2|z - z'|}{|i - z| \cdot |i - z'|}\,, \qquad z, z' \in \mathbb{C} - \{i\}. \tag{14}$$

Let W be the half-plane $\mathrm{Re}(w) > -\frac{1}{2}$. We claim that, for some $\mu_0 \in (0,1)$ and M,

$$z \in \mathbb{D}, \quad \mu \in [0,\mu_0] \quad \Rightarrow \quad U\big(F_\mu(z)\big) \in W, \quad \big|U\big(F_\mu(z)\big) - U(z)\big| \le M. \tag{15}$$

By the general theory ([7] p. 100), the function $(\mu, z) \mapsto F_\mu(z)$ is C^∞ where finite. Because each mapping F_μ has second-order contact with the identity at i, it then follows from Taylor's theorem that, for some $a \in (0,1]$, one has

$$\big|F_\mu(z) - z\big| < \frac{|z-i|^2}{8}, \qquad \mu \in [0, \tfrac{1}{2}], \quad |z-i| < a.$$

Let $A = \{z \in \mathbb{D} : |z-i| < a\}$ and $B = \{z \in \mathbb{D} : |z-i| \ge a\}$. If $z \in A$ and $\mu \in [0,\frac{1}{2}]$, then the conclusions in (15) hold with $M = \frac{1}{2}$, for $U(z) \in \mathbb{H}$ and, by (14),

$$\big|U\big(F_\mu(z)\big) - U(z)\big| \le \frac{2 \cdot |z-i|^2/8}{|z-i|\big(|z-i| - |z-i|^2/8\big)} < \frac{1/4}{1-1/8} < \frac{1}{2}.$$

The argument for points in B uses the convergence $\sup_{\mathbb{D}} |F_\mu - F_0| \to 0$ from Lemma 20. Equation (9) shows that $F_0(B)$ is contained in some set $\{w \in \mathbb{D} : |w-i| > \delta\}$. In view of the uniform continuity of U in the set $|w-i| > \delta/2$ evident from (14), one can therefore use Lemma 20 to choose $\mu_0 \in (0,\frac{1}{2}]$ so that $|F_\mu(z) - F_0(z)| < \delta/2$ and $|U(F_\mu(z)) - U(F_0(z))| < \frac{1}{2}$ whenever $z \in B$ and $\mu \in [0,\mu_0]$. For such z and μ, the point $U(F_\mu(z))$ is in W, for $U(F_0(z)) \in U(\mathbb{D}) = \mathbb{H}$, and

$$\begin{aligned}\big|U\big(F_\mu(z)\big) - U(z)\big| &\le \big|U\big(F_\mu(z)\big) - U\big(F_0(z)\big)\big| + \big|U\big(F_0(z)\big)\big| + |U(z)| \\ &< \frac{1}{2} + \frac{2}{\delta} + \frac{2}{a}.\end{aligned}$$

In all, (15) holds with this value μ_0 and $M = 1/2 + 2/\delta + 2/a$.

Let μ_0 and M be as in (15). A computation shows that $U \circ T_{-\alpha} \circ U^{-1}$ is multiplication by $(1-\alpha)/(1+\alpha)$ and that $U^{-1}(W)$ is the disk $|z+i| < 2$. Since $|U(z) - U(z')| \ge |z-z'|/8$ for all z, z' in that disk by (14), one also sees that $|U^{-1}(w) - U^{-1}(w')| \le 8|w-w'|$ for all $w, w' \in W$. It follows that

$$\begin{aligned}\big|F_{\mu\alpha}(z) - z\big| &= \left|U^{-1}\left(\frac{1-\alpha}{1+\alpha} \cdot \big(U \circ F_\mu \circ T_\alpha\big)(z)\right) - U^{-1}\left(\frac{1-\alpha}{1+\alpha} \cdot \big(U \circ T_\alpha\big)(z)\right)\right| \\ &\le 8(1-\alpha)\Big|\big(U \circ F_\mu \circ T_\alpha\big)(z) - \big(U \circ T_\alpha\big)(z)\Big| \le 8(1-\alpha)M\end{aligned}$$

when $z \in \mathbb{D}$ and $\mu \in [0,\mu_0]$, for multiplication by $(1-\alpha)/(1+\alpha)$ takes W into itself. Thus (d) in Lemma 21 holds with $K = 8M$. □

Step 3. The deduction of Lemma 17 from Lemma 21 is technical but not subtle. As in Lemma 17, let $C > 0$ and $\zeta \in \partial\mathbb{D}$, and let γ be a continuous, positive function in $[0, 1)$ such that $(\gamma(r) - 1)/(1 - r) \to \infty$ as $r \to 1$. The lemma asserts that, for all sufficiently small $\epsilon > 0$, there is a locally injective, holomorphic function f in a neighborhood of $\mathrm{clos}(\mathbb{D})$ such that

(i) $f(\mathbb{D}) \subseteq \mathbb{D}$,

(ii) $\frac{1}{2}[\![Sf]\!](z) \leq \gamma(|z|) - C\epsilon^2$ for all $z \in \mathbb{D}$,

(iii) $|f(z) - z| < \epsilon$ for all $z \in \mathbb{D}$,

(iv) $f(z) = f(\zeta')$ for some $z \in \mathbb{D}$ with $|z - \zeta| < 2\epsilon$ and some $\zeta' \in \partial\mathbb{D}$.

It suffices to prove this when $\zeta = -i$, for if f meets the requirement in that case then the function $z \mapsto i\zeta \cdot f(z/(i\zeta))$ meets them for an arbitrary point $\zeta \in \partial\mathbb{D}$.

Let μ_0 and K be as in Lemma 21, and let $r_0 \in (0, 1)$ be such that

$$\gamma(r) \geq 1 + 16K\sqrt{C}\,(1 - r), \qquad r \in [r_0, 1). \tag{16}$$

Assuming that $\epsilon > 0$ is less than or equal to the minimum of

$$\frac{\mu_0}{\sqrt{C}}, \quad 2K(1 - r_0)^2, \quad \frac{2}{1 + C} \cdot \min\{\gamma(r) : r \in [0, r_0]\}, \quad \frac{1}{2},$$

we set $\mu = \epsilon\sqrt{C}$ and $\alpha = 1 - \epsilon/(2K)$ and modify $F_{\mu\alpha}$ to produce a function f having the required properties.

The constraint $\epsilon \leq \mu_0/\sqrt{C}$ implies that $\mu \leq \mu_0$, so that Lemma 21 applies. Part (a) of the lemma asserts $F_{\mu\alpha}$ fails to be injective in $\{z \in \mathbb{D} : |z + i| < \epsilon/2\}$, part (d) that $|F_{\mu\alpha}(z) - z| < \epsilon/2$ for all $z \in \mathbb{D}$, and parts (b) and (c) that

$$\tfrac{1}{2}[\![S(F_{\mu\alpha})]\!](z) \leq \begin{cases} \epsilon/2 & \text{if } |z| \leq 1 - \sqrt{\epsilon/(2K)} \\[2ex] 1 - C\epsilon^2 + 16K\sqrt{C}\big(1 - |z|\big) & \text{for all } z \in \mathbb{D}. \end{cases}$$

From the latter bound and (16), it is clear that if $|z| \in [r_0, 1)$ then

$$\tfrac{1}{2}[\![S(F_{\mu\alpha})]\!](z) \leq \gamma\big(|z|\big) - C\epsilon^2. \tag{17}$$

The same conclusion holds by a different argument if $|z| \leq r_0$. For such z, the bound $\frac{1}{2}[\![S(F_{\mu\alpha})]\!](z) \leq \epsilon/2$ applies by virtue of the second constraint $\epsilon \leq 2K(1 - r_0)^2$ on ϵ, and $(1 + C)\epsilon/2 \leq \gamma(|z|)$ by virtue of the third. Because $\epsilon \leq \frac{1}{2}$, one has $\epsilon/2 \leq (1 + C)\epsilon/2 - C\epsilon^2$, and (17) follows.

Let $f(z) = (1-\epsilon/2)\cdot F_{\mu\alpha}(\rho z)$, where $\rho \in (0,1)$ is yet to be determined. This mapping is holomorphic and locally injective in the disk $|z| < 1/\rho$, and it satisfies (i) since

$$\big|f(z)\big| \le (1-\epsilon/2)\Big(\big|F_{\mu\alpha}(\rho z) - \rho z\big| + |\rho z|\Big) < (1-\epsilon/2)\big(\epsilon/2+\rho\big) < 1-\epsilon^2/4$$

for all $z \in \mathbb{D}$. It also satisfies (ii), for $|(Sf)(z)| = |\rho^2(SF_{\mu\alpha})(\rho z)|$ by the Schwarzian chain rule (3), and in view of the maximum principle and (17) it follows that

$$\big|(Sf)(z)\big| \le \big|(SF_{\mu\alpha})(\rho z)\big| \le \max_{|w|=|z|}\big|(SF_{\mu\alpha})(w)\big| \le 2\cdot\frac{\gamma(|z|)-C\epsilon^2}{(1-|z|^2)^2}\,, \qquad z \in \mathbb{D}.$$

Using the triangle inequality and the bound $|F_{\mu\alpha}(\rho z) - \rho z| < \epsilon/2$, one also sees that $|f(z) - z| < \epsilon$ for all $z \in \mathbb{D}$, so that (iii) holds, if $\rho > 1-\epsilon^2/4$.

It remains to show that (iv) holds when ρ is sufficiently near one. As noted above, there are points $z_1 \neq z_2$ in the set $A = \{z \in \mathbb{D} : |z+i| < \epsilon/2\}$ that map to the same image under $F_{\mu\alpha}$. Since $\rho z_1, \rho z_2 \in A$ when ρ is sufficiently near one, the mapping f also fails to be injective in A for such ρ. Assertion (iv) follows from that property and (iii). Indeed, let U and V be the sets consisting of all $z \in \mathbb{D}$ with $|z+i| < 2\epsilon$ and $|z+i| < 4\epsilon$, respectively. Since $|f-\mathrm{id}| < \epsilon$ throughout U and $|f-\mathrm{id}| \le \epsilon$ throughout ∂V, the triangle inequality shows that $f(U)$ is disjoint from $\{f(w) : w \in \partial V,\ |w+i| = 4\epsilon\}$. If (iv) fails, then $f(U)$ is disjoint from the rest of $f(\partial V)$, also, and since $f(U)$ is connected it follows from the argument principle that $f|_V$ attains every value in $f(U)$ the same number of times. Because $A \subseteq U$, that number is at least two. On the other hand, if $z_0 = -(1-2\epsilon)i$ then by Rouché's theorem $f|_V$ attains every value in the disk $|z-z_0| < \epsilon$ exactly once, and since that disk includes $f(z_0)$ it also includes $f(z)$ when $z \in U$ is near z_0. These two observations are contradictory, and the proof of Lemma 17 is complete.

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